

# ON VOLUME PRESERVING COMPLEX STRUCTURES ON REAL TORI

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**ABSTRACT.** A basic problem in the classification theory of compact complex manifolds is to give simple characterizations of complex tori. It is well known that a compact Kähler manifold  $X$  homotopically equivalent to a complex torus is biholomorphic to a complex torus.

The question whether a compact complex manifold  $X$  diffeomorphic to a complex torus is biholomorphic to a complex torus has a negative answer due to a construction by Blanchard and Sommese.

Their examples have however negative Kodaira dimension, thus it makes sense to ask the question whether a compact complex manifold  $X$  with trivial canonical bundle which is homotopically equivalent to a complex torus is biholomorphic to a complex torus.

In this paper we show that the answer is positive for complex threefolds satisfying some additional condition, such as the existence of a non constant meromorphic function.

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## 1. INTRODUCTION

The Enriques-Kodaira classification of compact complex surfaces implies in particular that a compact complex surface homotopically equivalent to a complex torus of dimension 2 is biholomorphic to a complex torus of dimension 2. The corresponding result in dimension 1 was already known in the 19th century.

Surprisingly, the analogous result in dimension 3 is no longer true, as shown by Sommese using results of Blanchard ([So75], p.213, (E) after [Bl53]; see also [Ca02], Section 5, [Ca04], Section 7).

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*Date:* December 30, 2009.

The present work took place in the realm of the DFG Forschergruppe 790 "Classification of algebraic surfaces and compact complex manifolds". AMS Classification: 32Q55, 32J17, 32J18 .

Indeed there are countably many families of complex manifolds even diffeomorphic to a complex torus of dimension 3, which are not biholomorphic to a complex torus.

These are constructed as follows: let  $L$  be a line bundle on a curve  $C$ , generated by global sections (if  $C$  is an elliptic curve, it suffices that the degree of  $L$  be at least 2). Let  $s_1, s_2 \in H^0(C, L)$  be two sections without common zeros, so that  $s := (s_1, s_2)$  is a nowhere vanishing section of the rank two vector bundle  $L \oplus L$ . Identifying the fibre  $\mathbb{C}^2$  with the quaternions, one finds that  $s, is, js, ks$  yield four sections  $\in H^0(C, L \oplus L)$  giving an  $\mathbb{R}$ -basis over each point (hence the total space of  $L \oplus L$  is diffeomorphic to a product  $C \times \mathbb{R}^4$ ).

Defining  $X$  as the quotient of the total space of  $L \oplus L$  by the free abelian subgroup  $\mathbb{Z}^4$  generated by the four sections,  $X$  is then diffeomorphic to a torus, yet its canonical bundle  $K_X$  has the property that  $K_X = p^*(-2L)$  ([Ca04], Remark 7.3), in particular  $h^0(X, -K_X) = h^0(C, 2L)$  which, in the case where  $C$  is an elliptic curve, equals 2  $\deg(L) \geq 4$ . Hence  $X$  is not a complex torus, for which  $K_X$  is a trivial divisor.

It is now natural to ask which kind of additional conditions are sufficient to characterize complex tori as complex manifolds. The simplest among such conditions, under a weak Kähler assumption (Theorem 2.3) requires to have the same integral cohomology algebra. However, if one drops the Kähler condition, the problem becomes much more difficult and, so far, not much is known (see however some characterizations in [Ca95] and in [Ca04], especially Proposition 2.9).

The examples of Blanchard-Sommese lead one of the authors ([Ca04] (p.269)) to ask the following question

*Are there compact complex manifolds  $X$  with trivial canonical bundle  $K_X$  which are diffeomorphic but not biholomorphic to a complex torus?*

In response to this question we prove the following theorem as a corollary of more general results (Theorems 2.3, 3.1, 3.2, 4.1, 5.1):

**Theorem 1.1.** *Let  $X$  be a compact complex manifold subject to the following conditions.*

- (1)  *$X$  is homotopy equivalent to a complex torus of dimension 3;*
- (2)  *$X$  has a dominant meromorphic map to a compact complex analytic space  $Y$  of smaller dimension, i.e., with  $0 < \dim Y < 3$ ;*
- (3)  *$K_X \equiv 0$ , i.e.,  $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$ .*

*Then  $X$  is biholomorphic to a complex torus.*

We should remark that condition (2) is of course not a necessary condition for  $X$  to be a complex torus. However, the only known examples of threefolds which are homeomorphic but not biholomorphic to a complex torus are Blanchard-Sommese's examples and they are all fibred over elliptic curves, as one can see from the construction described above. So they satisfy conditions (1) and (2) (but not (3)).

As a special case of Theorem 1.1, we obtain

**Corollary 1.2.** *Let  $X$  be a compact complex manifold such that:*

- (i)  *$X$  is homotopy equivalent to a complex torus of dimension 3;*
- (ii)  *$X$  has a non-trivial map  $\alpha : X \rightarrow T$  to a positive dimensional complex torus  $T$ ;*
- (iii)  *$K_X \equiv 0$ .*

*Then  $X$  is biholomorphic to a complex torus (of dimension 3).*

This is a direct consequence of Theorem 1.1 using ([Ue75], Lemma 10.1 and Theorem 10.3) in case  $\dim \alpha(X) = 3$ .

**Corollary 1.3.** *Let  $X$  be a compact complex manifold such that*

- (1)  *$X$  is homotopy equivalent to a complex torus of dimension 3;*
- (2) *either  $a(X) > 0$ , i.e.,  $X$  has a non-constant meromorphic function, or the Albanese torus  $\text{Alb}(X)$  is non-trivial;*
- (3)  *$K_X \equiv 0$ .*

*Then  $X$  is biholomorphic to a complex torus (of dimension 3).*

The remaining case where  $X$  has no non-constant meromorphic function and also no meromorphic map to a surface without meromorphic functions seems difficult.

If however the tangent or the cotangent bundle have some sections, the situation gets amenable:

**Theorem 1.4.** *Let  $X$  be a smooth compact complex threefold with  $K_X \equiv 0$  homotopy equivalent to a torus. If  $h^0(T_X) \geq 3$  or if  $h^0(\Omega_X^1) \geq 3$ , then  $X$  is biholomorphic to a torus.*

*Acknowledgements.* The second author is grateful to the guest program of the Alexander von Humboldt Foundation and to the DFG Forschergruppe 790 ‘‘Classification of algebraic surfaces and compact complex manifolds’’, which made this collaboration possible.

## 2. PRELIMINARIES

We start with some notations. Let  $X$  be an irreducible compact complex space. Then  $a(X)$  denotes the algebraic dimension of  $X$  ([Ue75], Definition 3.2), the maximal number of algebraically independent meromorphic functions. If  $a(X) = \dim X$ , then  $X$  is called a Moishezon manifold.

We also recall that for a compact complex manifold  $X$ , the Albanese torus of  $X$  is the complex torus defined by

$$\text{Alb}(X) = H^0(X, d\mathcal{O}_X)^\vee / \Lambda ,$$

where  $\Lambda$  is the minimal closed complex Lie group containing  $\text{Im}(H_1(X, \mathbb{Z}) \rightarrow H^0(X, d\mathcal{O}_X)^\vee)$ .

We have then the Albanese morphism  $\text{alb}_X : X \rightarrow \text{Alb}(X)$  (see [Ue75], p.101–104), assigning to each point  $x$  the class of the linear functional  $\int_{x_0}^x$  on  $H^0(X, d\mathcal{O}_X)$ , obtained integrating on a path from  $x_0$  to  $x$ .

**Proposition 2.1.** *Let  $f : X \rightarrow Y$  be a surjective morphism with connected fibres from a compact (connected) complex manifold  $X$  with  $\pi_1(X) \simeq \mathbb{Z}^k$  to a complex manifold  $Y$ . Let  $F$  be a general fibre of  $f$ . Then there exists an exact sequence of groups*

$$0 \longrightarrow A \longrightarrow \pi_1(X) \simeq \mathbb{Z}^k \xrightarrow{f_*} \pi_1(Y) \longrightarrow 0 ,$$

*where  $A$  contains  $\text{Im}(\pi_1(F) \rightarrow \pi_1(X))$  as a finite index subgroup. In particular,  $\pi_1(Y)$  is a finitely generated abelian group of rank  $\leq k$  and there is an inequality of Betti numbers*

$$b_1(F) + b_1(Y) \geq k = b_1(X) .$$

*Proof.* The following proof is very close to [No83], Lemma 1.5 and [CKO03], Lemma 3.

Let  $F$  be a general fibre of  $f$  and  $U \subset Y$  be the maximal Zariski open subset such that  $f$  is smooth over  $U$ . Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G & \longrightarrow & \pi_1(f^{-1}(U)) & \xrightarrow{(f_U)_*} & \pi_1(U) & \longrightarrow & 1 \\
\downarrow & & \downarrow b_F & & \downarrow b & & \downarrow c & & \downarrow \\
1 & \longrightarrow & \text{Ker } f_* & \longrightarrow & \pi_1(X) \simeq \mathbb{Z}^k & \xrightarrow{f_*} & \pi_1(Y) & \longrightarrow & 1
\end{array}$$

Here  $G = \text{Im}(\pi_1(F) \rightarrow \pi_1(f^{-1}(U)))$ . Since  $b$  is surjective, the snake lemma yields the following exact sequence:

$$\text{Ker } b \rightarrow \text{Ker } c \rightarrow \text{Coker } b_F \rightarrow 0.$$

Thus

$$\text{Ker } c / (f_U)_*(\text{Ker } b) \simeq \text{Coker } b_F = \text{Ker } f_* / \text{Im } b_F.$$

Since  $\text{Ker } f_* \subset \mathbb{Z}^k$ , it follows that

$$\text{Ker } c / (f_U)_*(\text{Ker } b) \simeq \text{Coker } b_F$$

is a finitely generated abelian group. On the other hand, each  $[\gamma] \in \text{Ker } c$  is represented by the product of conjugates of elements represented by a closed circle  $\gamma$  contained in  $D \cap (Y \setminus U)$  with a base point  $x$ , where  $D \simeq \Delta$  is a small disk on  $Y$  transversal to  $Y \setminus U$  in a point  $s_0$ .

Since however  $\pi_1(Y)$  is abelian, we see that  $\text{Ker } c$  is generated by such elements.

For each such element take a small disk  $\tilde{D} \simeq \Delta$  in  $X$  such that  $f(\tilde{D}) = D$ , and let  $d$  be the degree of the finite branched cover  $\tilde{D} \rightarrow D$ .

The preimage of  $\gamma$  in  $\tilde{D}$  is a closed circle  $\tilde{\gamma}$  such that  $f(\tilde{\gamma}) = d\gamma$ . Thus  $\text{Ker } c / (f_U)_*(\text{Ker } b)$ , is a torsion group. Hence  $\text{Coker } b_F$  is a finite abelian group. The last statement is clear from the fact that  $\pi_1(X)$ ,  $\pi_1(Y)$  and  $A$  are all abelian.  $\square$

From [Ca04], Proposition 2.9 (see also [Ca95] Corollary C, [Ca02], Proposition 4.8), we cite the following

**Theorem 2.2.** *Let  $X$  be a compact complex manifold of dimension  $n$  such that*

- (1) *The cohomology ring  $H^*(X, \mathbb{Z})$  is isomorphic to the cohomology ring of the  $n$ -dimensional complex torus.*
- (2)  *$H^0(X, d\mathcal{O}_X) = n$ , i.e., there are exactly  $n$  linearly independent  $d$ -closed holomorphic 1-forms.*

*Then  $X$  is biholomorphic to a complex torus.*

If  $X$  is bimeromorphically equivalent to a Kähler manifold, our main problem is easily answered.

**Theorem 2.3.** *Let  $X$  be a compact complex manifold such that*

- (1) *The cohomology ring  $H^*(X, \mathbb{Z})$  is isomorphic to the cohomology ring of the  $n$ -dimensional complex torus (for instance,  $X$  is homotopy equivalent to a complex torus of dimension  $n$ ).*

- (2)  $X$  is in the Fujiki class  $\mathcal{C}$ , i.e.,  $X$  is bimeromorphic to a compact Kähler manifold.

Then  $X$  is biholomorphic to a complex torus of dimension  $n$ .

*Proof.* We apply Theorem 2.2 to our  $X$ . The first condition in Theorem 2.2 holds by assumption. In particular,  $b_1(X) = 2n$ . As  $X$  is in class  $\mathcal{C}$ , every holomorphic form is  $d$ -closed and the Hodge decomposition holds for  $X$  ([Fj78], Corollary 1.7). Thus the second condition in Theorem 2.2 also holds and an application of Theorem 2.2 implies the result.  $\square$

A special case of Theorem 2.3 is

**Corollary 2.4.** *A Moishezon manifold  $X$  homotopy equivalent to a complex torus of dimension  $n$  is biholomorphic to an abelian variety.*

Recall that a compact complex manifold is said to be a *Moishezon* manifold if the algebraic dimension is maximal:  $a(X) = \dim X$ .

### 3. COMPLEX TORUS BUNDLES OVER A COMPLEX TORUS

In this section we prove two general results on submersions of special manifolds (Theorems 3.1, 3.2). These results are used in our proof of our Main Theorem 1.1. The crucial point in both results is that we do *not* assume the total space  $X$  to be Kähler.

**Theorem 3.1.** *Let  $f : X \rightarrow Y$  be a holomorphic submersion with connected fibres between compact (connected) complex manifolds and assume:*

- (1)  $X$  has complex dimension  $n + m$  and trivial canonical divisor  $K_X \equiv 0$ ;
- (2)  $Y$  has complex dimension  $m$  and also  $K_Y \equiv 0$ ;
- (3) every fiber  $X_y$  ( $y \in Y$ ) is Kähler;
- (4) the monodromy action of  $\pi_1(Y)$  on  $H^n(X_y, \mathbb{Z})$  is trivial.

Then all the fibres  $X_y$  are biholomorphic, and  $f$  is a holomorphic fibre bundle.

*Proof.* By (4),  $R^n f_* \mathbb{Z}_X$  is not only locally constant but also *globally constant* on  $Y$ . Thus, for the  $\mathbb{Z}_Y$  dual local system, we have

$$(R^n f_* \mathbb{Z}_X)^* \simeq H_n(X_b, \mathbb{Z})_f \times Y.$$

Here  $b \in Y$  is any base point and  $H_n(X_b, \mathbb{Z})_f$  denotes the free part of  $H_n(X_b, \mathbb{Z})$ . The same abbreviation will be applied for other points  $y \in Y$ . Let

$$\gamma_{1,b}, \dots, \gamma_{k,b}$$

be a basis of  $H_n(X_b, \mathbb{Z})_f$  and let

$$\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$$

be the corresponding flat basis of  $(R^n f_* \mathbb{Z}_X)^*$  over  $Y$ .

Then the elements  $\tilde{\gamma}_{i,y}$  ( $1 \leq i \leq k$ ) form a free basis of  $H_n(X_y, \mathbb{Z})_f$  for each  $y \in Y$ .

Now, following Fujita [Fu78], p. 780-781, we construct a family of holomorphic  $n$ -forms on the fibres, say  $\{\varphi_y\}_{y \in Y}$ , which varies holomorphically with respect to  $y \in Y$ .

Since

$$\omega_{X/Y} = \mathcal{O}(K_X) \otimes f^* \mathcal{O}(K_Y)^\vee \simeq \mathcal{O}_X,$$

we obtain that

$$f_*(\omega_{X/Y}) \cong \mathcal{O}_Y.$$

We are done by the exact sequence

$$0 \rightarrow f^*(\Omega_Y^1) \rightarrow \Omega_X^1 \rightarrow \Omega_{X|Y}^1 \rightarrow 0$$

and since by definition

$$\omega_{X/Y} := \det(\Omega_{X|Y}^1) = \Lambda^n(\Omega_{X|Y}^1).$$

Hence a global generator of  $f_*(\omega_{X/Y}) \cong \mathcal{O}_Y$  gives the desired family of holomorphic  $n$ -forms on the fibres, yielding a nowhere vanishing form on each fibre.

Note that  $\varphi_y$  is  $d$ -closed being a top holomorphic form.

Now we consider the *non-projectivized, global* period map:

$$\tilde{p}_Y : Y \rightarrow \mathbb{C}^k ; y \mapsto \left( \int_{\tilde{\gamma}_{i,y}} \varphi_y \right)_{i=1}^k .$$

This map is holomorphic by a fundamental result of Griffiths. Indeed, to be able to apply [Gr68], Theorem (1.1), we need that the fibres  $X_y$  are Kähler but we do not need that the total space  $X$  be Kähler.

On the other hand, since  $Y$  is compact, the global *holomorphic* functions on  $Y$  are constant. Thus all functions

$$y \mapsto \int_{\tilde{\gamma}_{i,y}} \varphi_y$$

are constant on  $Y$ . Hence the usual period map  $p_Y : Y \rightarrow \mathbb{P}^{k-1}$ , which is just the projectivization of the target domain  $\mathbb{C}^k$  of  $\tilde{p}_Y$ , is also constant as well.

As all the fibers  $X_y$  ( $y \in Y$ ) are compact Kähler manifolds with trivial canonical class, the local Torelli Theorem holds for them, i.e., the period map from the Kuranishi space to the period domain is injective (see e.g. [GHJ03], p. 109, Theorem 16.9; the proof given there is written only for Calabi-Yau 3-folds, but the proof in the general case is exactly the same).

Since  $p_Y$  is constant and  $Y$  is connected, it follows that all the fibres  $X_y$  are biholomorphic. Hence  $f$  is locally analytically trivial by the fundamental result of Grauert-Fischer (or by Kuranishi's theorem). This concludes the proof.  $\square$

**Theorem 3.2.** *Let  $f : X \rightarrow Y$  be a holomorphic submersion with connected fibres between compact (connected) complex manifolds and assume:*

- (1)  *$X$  is homotopy equivalent to a complex torus of dimension  $n + m$ ;*
- (2)  *$Y$  is a complex torus of dimension  $m$ ;*
- (3) *some fibre  $X_y$  is biholomorphic to a complex torus.*

*Then  $f : X \rightarrow Y$  is a principal holomorphic torus bundle and  $X$  is biholomorphic to a complex torus.*

*Proof.* By [Ca04], Theorem 2.1, every fibre  $X_y$  is isomorphic to a complex torus of dimension  $n$ . Let  $F = X_y$  be one of the fibres of  $f$ . Since  $\pi_2(Y) = 0$ , we have the following exact sequence

$$0 \rightarrow \pi_1(F) \simeq \mathbb{Z}^{2m} \rightarrow \pi_1(X) \rightarrow \pi_1(Y) \simeq \mathbb{Z}^{2n} \rightarrow 0 .$$

Since  $\pi_1(X) \simeq \mathbb{Z}^{2(n+m)}$  by (1), this sequence splits and  $\pi_1(Y)$  acts on  $\pi_1(F)$  as the identity. Then, by Theorem 3.1,  $f$  is a holomorphic fibre bundle. In particular, the Kodaira-Spencer map

$$T_{Y,y} \rightarrow H^1(X_y, T_{X_y})$$

of  $f$  is zero at every point  $y \in Y$ . Then, by [Ca04], Proposition 3.2 and its proof,  $f$  is a principal fibre bundle with structure group  $F$ , i.e., a fibre bundle whose transition functions are given by translations by local holomorphic sections of  $F$  over  $Y$ . We want to show that they can actually be chosen to be locally constant.

To verify this, we follow [BHPV04], p.196. Set  $\Gamma = H_1(F, \mathbb{Z}) \simeq \mathbb{Z}^{2n}$ .

Consider the following commutative diagram of exact sequences of abelian sheaves on  $Y$ :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma = \Gamma_Y & \longrightarrow & \mathbb{C}_Y^n & \longrightarrow & F_Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma = \Gamma_Y & \longrightarrow & \mathcal{O}_Y^n & \longrightarrow & \mathcal{F}_Y & \longrightarrow & 0 \end{array}$$

Here  $F_Y$  is the abelian sheaf of locally constant sections with values in  $F$  and  $\mathcal{F}_Y$  is the abelian sheaf of holomorphic sections with values in  $F$ .

Taking the corresponding cohomology sequences yields the diagram

$$\begin{array}{ccccc} H^1(Y, \mathbb{C}^n) & \xrightarrow{\gamma} & H^1(Y, F_Y) & \longrightarrow & H^2(Y, \Gamma) \\ \beta_1 \downarrow & & \beta_2 \downarrow & & \downarrow \\ H^1(Y, \mathcal{O}_Y^n) & \xrightarrow{\alpha} & H^1(Y, \mathcal{F}_Y) & \xrightarrow{c} & H^2(Y, \Gamma) \end{array}$$

Let  $\eta \in H^1(Y, \mathcal{F}_Y)$  be the class representing the principal holomorphic bundle structure of  $f : X \rightarrow Y$ . Set  $\epsilon = c(\eta)$ . Note that  $f$  is topologically trivial, since the exact sequence of the fundamental group splits trivially. Thus  $\epsilon = 0$  and therefore  $\eta = \alpha(\eta_1)$  for some  $\eta_1 \in H^1(Y, \mathcal{O}_Y^n)$ . Since  $Y$  is Kähler, the map  $\beta_1$  is the one induced by the natural projection under the Hodge decomposition

$$H^1(Y, \mathbb{C}) = H^1(\mathcal{O}_Y) \oplus H^0(\Omega_Y^1).$$

In particular,  $\beta_1$  is surjective. Thus  $\eta_1 = \beta_1(\eta_2)$  for some  $\eta_2 \in H^1(Y, \mathbb{C}^n)$ . Hence

$$\eta = \alpha\beta_1(\eta_2) = \beta_2\gamma(\eta_2) = \beta_2(\eta_3),$$

where  $\eta_3 = \gamma(\eta_2) \in H^1(Y, F_Y)$ . This means that the transition functions defining the principal bundle structure  $f : X \rightarrow Y$  can be chosen to be *locally constant*.

We can now use two arguments, here is the first.

Let  $Y = \cup_{i \in I} U_i$  be a sufficiently small open covering of  $Y$  with trivializations

$$\varphi_i : X_{U_i} \simeq F \times U_i,$$

such that the transition functions  $\varphi_i^{-1} \circ \varphi_j$  are all constant on  $U_i \cap U_j$ . Let  $\tau_Y$  be a standard Kähler form on  $Y$  and  $\tau_F$  be a standard Kähler form on  $F$ .

Set  $\tau_i = \tau_Y|_{U_i}$ . Then  $\tilde{\tau}_i := \varphi_i^*(\tau_i \wedge \tau_F)$  gives a Kähler form on  $X_{U_i}$ . As  $\varphi_i^{-1} \circ \varphi_j$  is a translation by some *constant* element of  $F$  over  $U_i \cap U_j$ , it follows that  $\tilde{\tau}_i = \tilde{\tau}_j$  on  $X_{U_i} \cap X_{U_j}$ . Hence  $\{\tilde{\tau}_i\}_{i \in I}$  defines a global Kähler form on  $X$ . In particular  $X$  is Kähler and therefore  $X$  is biholomorphic to a complex torus by Theorem 2.3.

Alternatively, one immediately sees that the fact that the transition functions defining the principal bundle structure  $f : X \rightarrow Y$  are locally constant implies that

the space of closed holomorphic 1-forms  $H^0(X, d\mathcal{O}_X)$  has dimension at least  $n$ , and we can apply Theorem 2.2.  $\square$

#### 4. A CHARACTERIZATION OF COMPLEX TORI - THE CASE FIBRED BY CURVES

The goal of this section is to prove the following

**Theorem 4.1.** *Let  $X$  be a compact complex manifold subject to the following conditions.*

- (1)  $X$  is homotopy equivalent to a complex torus of dimension  $m + 1$ ;
- (2) there is a dominant meromorphic map  $f : X \dashrightarrow Y$  to a compact complex manifold  $Y$  with  $\dim Y = m$ ;
- (3)  $m \leq 2$  or  $Y$  is Moishezon with  $\kappa(Y) \geq 0$ ;
- (4)  $K_X \equiv 0$ .

*Then  $X$  is biholomorphic to a complex torus of dimension  $m + 1$ .*

In the rest of this section, we shall prove Theorem 4.1 and always assume the situation of Theorem 4.1. Take a resolution of indeterminacies  $\nu : \tilde{X} \rightarrow X$  of  $f$ , yielding a surjective morphism

$$\tilde{f} : \tilde{X} \rightarrow Y$$

By considering the Stein factorization we may assume that  $\tilde{f}$  has connected fibers; loosely speaking  $f$  has connected fibers. In case  $Y$  is Moishezon, we may replace  $Y$  by a suitable birational model and therefore may assume  $Y$  to be projective. Finally  $F$  will always denote a smooth fibre of  $\tilde{f}$ .

**Lemma 4.2.** (1) *If  $\kappa(Y) \geq 0$ , all smooth fibers of  $\tilde{f}$  are isomorphic to a single elliptic curve, say  $E$ , and  $\kappa(Y) = 0$ .*  
(2) *If moreover  $Y$  is projective, then  $Y$  is birational to an abelian variety of dimension  $m$ . More precisely, the Albanese map  $a : Y \rightarrow \text{Alb } Y$  is a birational surjective morphism.*

*Proof.* (1) Since  $K_{\tilde{X}}$  is effective, the fiber  $F$  has genus  $g(F) \geq 1$ . Then by [Ue87], Theorem 2.1,  $F$  must actually be an elliptic curve. Moreover by [Ue87], Theorem 2.2,

$$0 = \kappa(X) = \kappa(\tilde{X}) \geq \max(\kappa(Y), \text{var}(\tilde{f})) \geq 0,$$

where  $\text{var}(\tilde{f})$  denotes the variation of  $\tilde{f}$ . Thus  $\kappa(Y) = 0$  and  $\text{var}(\tilde{f}) = 0$  and the first assertion is proven.

(2) For the second assertion assume now that  $Y$  is projective. Note that  $\pi_1(\tilde{X}) \simeq \mathbb{Z}^{2(m+1)}$ , since  $X$  is homotopy equivalent to a complex torus of dimension  $m + 1$ . Thus, applying Proposition 2.1,

$$\pi_1(Y) \simeq \mathbb{Z}^n$$

(up to torsion) for some integer  $n$  such that  $2m \leq n \leq 2(m + 1)$ . Since  $Y$  is a projective manifold, Hodge decomposition gives either  $n = 2m$  or  $n = 2(m + 1)$  and  $h^1(\mathcal{O}_Y) = m$  or  $h^1(\mathcal{O}_Y) = m + 1$ . Since  $\kappa(Y) = 0$ , a fundamental result due to Kawamata ([Ka81], main theorem) yields

$$h^1(\mathcal{O}_Y) = m (= \dim Y,)$$



and also the birationality of the Albanese morphism  $a : Y \longrightarrow \text{Alb } Y$ . This completes the proof.  $\square$

**Lemma 4.3.** *Assume that  $m \leq 2$ . Then  $X$  is a complex torus or the following two statements hold (recall that we assume  $f$  to have connected fibers).*

- (1) *All smooth fibres are isomorphic to a fixed elliptic curve, say  $E$ ;*
- (2)  *$Y$  is bimeromorphic to a complex torus of dimension 2. More precisely, the Albanese map  $a : Y \rightarrow \text{Alb } Y$  is a bimeromorphic surjective morphism.*

*Proof.* When  $m = 1$ , we have  $\dim X = 2$ . Then by classification  $X$  is a complex torus since  $K_X = \mathcal{O}_X$  and  $b_1(X) = 4$ . So from now we shall assume that  $m = 2$ . We may also assume that  $Y$  is a minimal surface.

Suppose first that  $\kappa(Y) \geq 0$ , hence  $\kappa(Y) = 0$  by 4.2(1), and  $F$  is a fixed elliptic curve. If  $Y$  would not be complex torus, then  $b_1(Y) \leq 3$  by classification. Then however

$$b_1(F) + b_1(Y) \leq 5 < 6 = b_1(X) = b_1(\tilde{X}) ,$$

a contradiction to Proposition 2.1.

It remains to consider the case where  $\kappa(Y) = -\infty$ . If in addition  $Y$  is Kähler, then  $Y$  is projective (rational or birationally ruled) by classification. We also have  $b_1(Y) \leq 2$  by the fact that  $\pi_1(Y)$  is abelian (Proposition 2.1). Then  $b_1(F) \geq 4$  for the general fibre  $F$  of  $\tilde{f} : \tilde{X} \rightarrow Y$ , again by Proposition 2.1. Therefore  $g(F) \geq 2$ , where  $g(F)$  is the genus of the curve  $F$ . Then we have a relative pluri-canonical map  $\hat{X} \dashrightarrow Z$  of  $\tilde{X}$  over  $Y$  ([Ue75], Theorem 12.1 and its proof). As  $Y$  is projective,  $Z$  is a projective 3-fold by the construction given there. Hence

$$a(X) = a(\tilde{X}) = a(Z) = 3$$

and we conclude that  $X$  is biholomorphic to a complex torus by Theorem 2.3 or by Corollary 2.4, and we are done.

If  $\kappa(Y) = -\infty$  and  $Y$  is not Kähler, then  $Y$  is a minimal surface of class VII. In particular,  $Y$  is not covered by rational curves and  $b_1(Y) = 1$ . Now observe that  $f$  is almost holomorphic in the sense that  $f$  is proper holomorphic over Zariski dense open subset of  $Y$ . Indeed, otherwise the exceptional locus of the resolution of indeterminacies  $\tilde{X} \rightarrow X$  dominates  $Y$ , so that  $Y$  would be dominated by a uniruled surface contradicting the assumption that  $Y$  is of class VII. Now  $f$  being almost holomorphic, the general fibre  $F$  of  $\tilde{f}$  is an elliptic curve by adjunction. Thus

$$b_1(F) + b_1(Y) = 3 < 6 = b_1(X) = b_1(\tilde{X}),$$

a contradiction to Proposition 2.1. This completes the proof.  $\square$

The upshot of the preceeding two lemmata is that we may assume  $Y$  to be a torus. In particular the meromorphic map  $f : X \dashrightarrow Y$  (from our *original*  $X$ ) is holomorphic and all smooth fibers are isomorphic to a fixed elliptic curve  $E$ .

**Lemma 4.4.** (1)  *$f$  is smooth in codimension 1, that is, the set of critical values of  $f$  is of codimension  $\geq 2$  on  $Y$ ;*  
(2)  *$f$  is equi-dimensional, or equivalently,  $f$  is a flat morphism.*

*Proof.* (1) (a) Let us first consider the case that  $Y$  is projective. Then we take a general complete intersection curve  $C$  on  $Y$ , i.e., a complete intersection of  $m - 1$  general hyperplanes of  $Y$ . So by Bertini's theorem,  $C$  is a smooth curve and  $X_C = f^{-1}(C)$  is a smooth surface. Let  $f_C : X_C \rightarrow C$  be the induced morphism; then

it suffices to show that  $f_C$  is a smooth morphism. By the adjunction formula, by  $K_X = \mathcal{O}_X$  and  $K_Y = \mathcal{O}_Y$ , we obtain

$$K_{X_C} = f_C^*(K_C),$$

i.e.,  $K_{X_C/C} = \mathcal{O}_{X_C}$ . Then the canonical bundle formula for an elliptic surface (see eg. [BHPV04], Page 213, Theorem 12.3) gives the smoothness of  $f_C$ .

(b) It remains to consider the case where  $\dim Y = 2$  and  $Y$  not projective. If  $a(Y) = 0$ , then  $Y$  has no complete curve and  $f$  is smooth in codimension 1. If  $a(Y) = 1$ , then the algebraic reduction  $a : Y \rightarrow C$  of  $Y$  is a smooth elliptic fibration over an elliptic curve  $C$  and all curves on  $Y$  are fibres of  $a$ . Thus the 1-dimensional part of the critical values form a normal crossing divisor and we can apply the canonical bundle formula ([Ue87], Theorem 2.4, or [Fu86], Theorem 2.15) to our elliptic 3-fold  $f : X \rightarrow Y$ . As a result, if the set of the critical values is *not* of codimension  $\geq 2$ , then there are fibres  $C_i$  ( $1 \leq i \leq k$ ) of  $a$  and positive integers  $n_i$  and  $M$  such that we have a bijection

$$|MK_X| \leftrightarrow |f^*(MK_Y + \sum_{i=1}^k n_i C_i)|.$$

This is however absurd, because the left hand side is an empty set by  $K_X = \mathcal{O}_X$ , but the right hand side is a non-empty set since  $K_Y = \mathcal{O}_Y$  and  $n_i > 0$ .

This completes the proof of (1).

(2) To begin with, notice that equi-dimensionality and flatness are equivalent,  $X$  and  $Y$  being smooth. We denote the union of all irreducible components of dimension  $\geq 2$  in the fibers of  $f$  by  $N_0$ . Assuming  $N_0 \neq \emptyset$  shall derive a contradiction. To do that, let

$$N = f^{-1}f(N_0).$$

First of all,  $N$  must be of pure codimension 1 in  $X$ . In fact, otherwise take a general small  $m$ -dimensional disk  $\Delta$  centered at a general point  $P$  of a 1-dimensional component of  $N$ . Then  $\Delta$  dominates  $Y$  at  $f(P)$  and  $f|_{\Delta} : \Delta \rightarrow Y$  is a generically finite surjective morphism around  $f(P)$  branched in codimension  $\geq 2$  on  $\Delta$ . However this is impossible by the purity of the branch loci. Thus  $N$  is a divisor.

Choose an irreducible component  $B$  of  $N$ .

By Hironaka's flattening theorem ([Hi75], main result), there is a successive sequence of blow-ups  $\mu : \hat{Y} \rightarrow Y$  such that the induced morphism

$$f_1 : X_1 := X \times_Y \hat{Y} \rightarrow \hat{Y}$$

is a flat morphism. Let

$$E'_i \ (1 \leq i \leq k)$$

be the exceptional divisors of  $\mu : \hat{Y} \rightarrow Y$ . Since flatness is preserved under base change, we may assume that  $\sum_{i=1}^k E'_i$  is a normal crossing divisor, possibly performing further blow-ups of  $\hat{Y}$ . Consider the normalization  $X_2 \rightarrow X_1$  of  $X_1$  and perform a resolution of singularities ([Hi77], main result) of  $X_2$ , say  $X_3 \rightarrow X_2$  and then finally take a resolution of indeterminacies (ibid.) of  $X \dashrightarrow X_3$ , say  $\pi : \hat{X} \rightarrow X$ . Let  $\hat{f} : \hat{X} \rightarrow \hat{Y}$  be the induced morphism.

Let  $E_j$  ( $1 \leq j \leq \ell$ ) be the exceptional divisors of  $\pi : \hat{X} \rightarrow X$  and  $\hat{B}$  be the proper transform of  $B$  on  $\hat{X}$ . Since  $B$  is of codimension 1 on  $X$ , necessarily  $\hat{B} \neq E_j$

for any  $j$ . On the other hand, the fact that we have flattened  $f$  means that  $\hat{f}(\hat{B})$  is one of the  $E'_i$ , say  $E'_1$ .

We are going to apply the canonical bundle formula for  $\hat{f}$  in [Ue87], Theorem 2.4 (or [Fu86], Theorem 2.15). Note that

$$K_{\hat{X}} = \sum_{j=1}^{\ell} a_j E_j$$

with every  $a_j > 0$  since  $K_X = \mathcal{O}_X$ . For the same reason,

$$K_{\hat{Y}} = \sum_{i=1}^k b_i E'_i$$

with every  $b_i > 0$ . As  $f$  is smooth in codimension 1, the discriminant divisor of  $\hat{f}$  is supported in  $\bigcup_{i=1}^k E'_i$ . Thus for a large multiple  $M > 0$ , we obtain

$$\begin{aligned} M \sum_{j=1}^{\ell} a_j E_j &= M K_{\hat{X}} \\ &= \hat{f}^*(M K_{\hat{Y}} + \sum_{i=1}^k c_i E'_i) + D_1 - D_2 = \hat{f}^*\left(\sum_{i=1}^k ((b_i + c_i) E'_i) + D_1 - D_2\right) \end{aligned}$$

where  $D_1$  is an effective divisor such that no multiple of  $D_1$  moves,  $D_2$  is an effective divisor such that  $\hat{f}(D_2)$  is of codimension  $\geq 2$ , and each  $c_i$  is a non-negative integer. Notice  $b_i + c_i > 0$  for all  $i$ . Moreover, by [Ue87], Theorem 2.4 (especially the statement (6) there), every element of  $|M K_X|$  is uniquely written, as a divisor, in the form of the sum of an element of

$$|\hat{f}^*\left(\sum_{i=1}^k (b_i + c_i) E'_i\right)| = \hat{f}^*\left|\sum_{i=1}^k (b_i + c_i) E'_i\right| = \{\hat{f}^*\left(\sum_{i=1}^k (b_i + c_i) E'_i\right)\}$$

and the divisor  $D_1 - D_2$ . Thus, we have the following equality as a *divisor*:

$$M \sum_{j=1}^{\ell} a_j E_j + D_2 = \hat{f}^*\left(\sum_{i=1}^k (b_i + c_i) E'_i\right) + D_1 .$$

In this equality of divisors, the prime divisor  $\hat{B}$  appears in the right hand side because  $\hat{f}(\hat{B}) = E'_1$ . But it does not appear in the left hand side, since - as already observed -  $\hat{B}$  is neither  $\pi$ -exceptional nor  $\hat{f}(\hat{B})$  is of codimension  $\geq 2$  in  $\hat{Y}$ . This contradiction concludes the equi-dimensionality of  $f$  and the flatness of  $f$  as well.  $\square$

**Lemma 4.5.** *The map  $f : X \rightarrow Y$  is smooth.*

*Proof.* Let  $y \in Y$  be any point of  $Y$  and suppose that the fiber  $X_y$  is singular. If  $X_y$  has a non-reduced component  $C$ , necessarily of dimension 1 by Lemma 4.4, we choose an  $m$ -dimensional general disk  $\Delta$  centered at a general non-reduced point  $P \in C$ . Then  $f|_{\Delta} : \Delta \rightarrow Y$  is a generically finite surjective morphism around  $f(P)$  whose branch locus in  $\Delta$  is of codimension  $\geq 2$  (since  $f$  is smooth in codimension 1), a contradiction to the purity of branch loci.

Hence  $X_y$  is reduced. Now take a local section  $D$  at a general point of  $X_y$ . Once

we have chosen  $D$ , we can describe the fibration by the Weierstrass equation locally near  $y$ :

$$y^2 = x^3 + a(t)x + b(t)$$

where  $a(t)$ ,  $b(t)$  are holomorphic functions around  $y$ . Then the critical locus of the original  $f$  around  $y$  is given by the equation

$$4a(t)^3 + 27b(t)^2 = 0.$$

In particular, it is of pure codimension 1 on  $Y$  unless it is empty. As  $f$  is smooth in codimension 1, it follows that the critical locus is empty, that is,  $f$  is smooth.  $\square$

Finally we can apply Theorem 3.2 to conclude Theorem 4.1.

Q.E.D. for Theorem 4.1.

**Remark 4.6.** In case  $\dim X = 3$ , Lemma 4.5 can be proved without using Weierstrass models in the following way. So we suppose that we already as shown in the first part of the proof of Lemma 4.5 without using the Weierstrass normal form that  $f$  can has finitely many singularities, say  $x_1, \dots, x_N$ . Choose a general holomorphic 1-form  $\omega$  on  $Y$ . Then  $f^*(\omega)$  vanishes exactly at  $x_1, \dots, x_N$ , hence  $c_3(\Omega_X^1) > 0$ . But  $c_3(X) = \chi_{\text{top}}(X) = 0$ , since  $X$  is homotopy equivalent to a torus.

## 5. A CHARACTERIZATION OF COMPLEX TORI - THE CASE FIBRED OVER A CURVE

In this section we shall prove the following:

**Theorem 5.1.** *Let  $X$  be a compact complex manifold such that*

- (1)  *$X$  is homotopy equivalent to a complex torus of dimension 3;*
- (2) *there is a dominant meromorphic map  $f : X \dashrightarrow Y$  to a smooth compact curve;*
- (3)  *$\mathcal{O}_X(K_X) \cong \mathcal{O}_X$ .*

*Then  $X$  is biholomorphic to a complex torus.*

Observe that in condition (2), taking the Stein factorization, we may assume  $f$  to have connected fibers.

We start with

**Lemma 5.2.** *Let  $X$  be a compact complex manifold subject to the assumptions in Theorem 5.1. Then  $X$  is biholomorphic to a complex torus of dimension 3 or  $Y$  is an elliptic curve,  $f$  is a holomorphic map and the general smooth fibre of  $f$  is a complex torus of dimension 2.*

*Proof.* By Proposition 2.1,  $\pi_1(Y)$  is an abelian group. Hence  $Y$  is either an elliptic curve or  $\mathbb{P}^1$ .

Consider first the case that  $Y$  is an elliptic curve, so  $f$  is holomorphic. Let  $F$  be a general fibre of  $f$ . Then  $K_F = \mathcal{O}_F$  by the adjunction formula. Since

$$b_1(F) + b_1(Y) \geq b_1(X) = 6$$

by Proposition 2.1, it follows from the classification of compact complex surfaces with  $\kappa = 0$  (see e.g. [BHPV04], p.244, Table 10) that  $b_1(F) = 4$  and  $F$  is a complex torus of dimension 2 so that we are done.

In case  $Y = \mathbb{P}^1$  let  $\tilde{f} : \tilde{X} \rightarrow Y$  be a resolution of indeterminacies of  $f$  and let  $F$  be a general fibre of  $\tilde{f}$ . Then  $F$  is smooth and  $\kappa(F) \geq 0$  by the adjunction formula.

If  $\kappa(F) \geq 1$ , then we can take a relative pluri-canonical map  $\varphi : X \dashrightarrow Z$  from  $X$  over  $Y$  ([Ue75], Theorem 12.1 and its proof). As  $Y$  is projective and  $Z$  is projective over  $Y$ , it follows that  $Z$  is projective. We have also  $\dim Z \geq 2$ . Thus  $X$  is biholomorphic to a complex torus by Theorem 4.1.

If  $\kappa(F) = 0$ , then  $b_1(F) \leq 4$  again by classification of compact complex surfaces with  $\kappa = 0$ . Then however

$$b_1(F) + b_1(Y) \leq 4 < 6 = b_1(X) = b_1(\tilde{X}),$$

contradicting Proposition 2.1.

This completes the proof.  $\square$

From now on we may assume that we have a surjective holomorphic map  $f : X \rightarrow Y$  over an elliptic curve  $Y$  with connected fibres.

The next two propositions of more topological nature will be applicable to many other situations:

**Proposition 5.3.** *Let  $X$  resp.  $Y$  be a topological space which is homotopy equivalent to a real torus  $A$  of real dimension  $N$  resp. homotopy equivalent to a real torus  $B$  of real dimension  $r$ . Let  $f : X \rightarrow Y$  be a continuous map which is dominant in the sense that  $f^* : H^r(Y, \mathbb{Z}) \rightarrow H^r(X, \mathbb{Z})$  is non-zero. Let  $u : \hat{Y} \rightarrow Y$  be the universal covering map and  $\hat{X} = X \times_Y \hat{Y}$  be the fibre product. Then  $\hat{X}$  is homotopy equivalent to a real torus of real dimension  $N - r$ , in particular we have  $H^{N-r}(\hat{X}, \mathbb{Z}) = \mathbb{Z}$ .*

*Proof.* Notice that we have natural isomorphisms  $\pi_1(X) \simeq H_1(X, \mathbb{Z})$  and  $\pi_1(Y) \simeq H_1(Y, \mathbb{Z})$  and - by our assumptions - they are isomorphic as abstract groups to  $\mathbb{Z}^N$  and  $\mathbb{Z}^r$ , respectively. Let us consider the homomorphism

$$f_* : \pi_1(X) \simeq \mathbb{Z}^N \rightarrow \pi_1(Y) \simeq \mathbb{Z}^r,$$

induced by  $f$ . Under the above isomorphisms, this homomorphism is the same as the homomorphism

$$f_* : H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^N \rightarrow H_1(Y, \mathbb{Z}) \simeq \mathbb{Z}^r.$$

The dual homomorphism

$$(f_1)^* : H^1(Y, \mathbb{Z}) \simeq \mathbb{Z}^r \rightarrow H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^N$$

is a part of the homomorphism of algebras given by the pullback

$$f^* : H^*(Y, \mathbb{Z}) \rightarrow H^*(X, \mathbb{Z}).$$

Again since  $X$  and  $Y$  are homotopy equivalent to real tori of dimensions  $N$  and  $r$  respectively, we know that

$$\wedge^r H^1(Y, \mathbb{Z}) \simeq H^r(Y, \mathbb{Z}) \simeq \mathbb{Z}, \quad \wedge^r H^1(X, \mathbb{Z}) \simeq H^r(X, \mathbb{Z}),$$

and the natural homomorphism

$$(f_r)^* : H^r(Y, \mathbb{Z}) \simeq \mathbb{Z} \rightarrow H^r(X, \mathbb{Z})$$

is simply  $\wedge^r (f_1)^*$ . As this is not zero by assumption, it follows that  $(f_1)^*$  is injective. Hence  $f_* : \pi_1(X) \rightarrow \pi_1(Y)$  is surjective up to a finite cokernel.

Now  $f$  factors through the finite unramified cover  $Y'$  of  $Y$  corresponding to  $f_*(\pi_1(X))$ , whence we may replace  $Y$  by  $Y'$  and assume that  $f_*$  is surjective.

We have then  $\text{Ker } f_* \simeq \mathbb{Z}^{N-r}$  and  $\pi_1(X)$  splits as

$$\pi_1(X) \simeq \mathbb{Z}^N \simeq \mathbb{Z}^r \oplus \mathbb{Z}^{N-r} \simeq \pi_1(Y) \oplus \text{Ker } f_*.$$

Hence under the universal covering maps  $u_B : \mathbb{R}^r \rightarrow B$  (corresponding to  $\pi_1(Y) \simeq \pi_1(B)$ ) and  $u_A : \mathbb{R}^N \rightarrow A$  (corresponding to  $\pi_1(X) \simeq \pi_1(A)$ ), it follows that  $X'$  is homotopy equivalent to

$$\mathbb{R}^N / \text{Ker } f_* \simeq \mathbb{R}^r \times (\mathbb{R}^{N-r} / \mathbb{Z}^{N-r}) \simeq \mathbb{R}^r \times T^{N-r} ,$$

where  $T^{N-r}$  is a real torus of dimension  $N - r$ . The result is now obvious.  $\square$

**Proposition 5.4.** *Let  $X$  be a compact complex manifold homotopy equivalent to a complex torus of dimension  $n + 1$  and let  $Y$  be an elliptic curve and let  $f : X \rightarrow Y$  be a surjective holomorphic map with connected fibres.*

*Then all analytic sets  $f^{-1}(y)$  ( $y \in Y$ ) are irreducible. Moreover, if  $Z$  is the reduction of a singular fiber and  $\tilde{Z}$  is a resolution of singularities of  $Z$ , then there is a real torus of real dimension  $2n$  and a surjective differentiable map  $\rho : \tilde{Z} \rightarrow T^{2n}$  such that the induced homomorphism*

$$\rho_* : \pi_1(\tilde{Z}) \rightarrow \pi_1(T^{2n})$$

*is surjective.*

*Proof.* As  $f$  is proper holomorphic and  $Y$  is compact, the critical values of  $f$  consist of finitely many points, say

$$B := \{b_1, b_2, \dots, b_k\} .$$

Let  $u : \mathbb{C} \rightarrow Y$  be the universal cover of  $Y$ . We consider the fiber product  $\hat{X} = X \times_Y \mathbb{C}$  and let  $\hat{f} : \hat{X} \rightarrow \mathbb{C}$  be the induced holomorphic map. The set of critical values of  $\hat{f}$  is  $\hat{B} = u^{-1}(B)$ . This is a discrete set of points of  $\mathbb{C}$ . By an appropriate choice of the origin in  $\mathbb{C}$ , we may assume that  $0 \notin \hat{B}$ . Then  $u^{-1}(u(0))$  forms a lattice  $\Lambda$  such that  $Y = \mathbb{C}/\Lambda$ . We choose generators of  $\Lambda$ , say  $v_1$  and  $v_2$ . Then each region

$$U_{n,m} := \{\alpha v_1 + \beta v_2 \mid n \leq \alpha < n+1, m \leq \beta < m+1\}$$

$(n, m \in \mathbb{Z})$  forms a fundamental domain for  $Y$ . We first take the following contractible graph

$$\Gamma_0 := \mathbb{R}v_1 \cup \bigcup_{k \in \mathbb{Z}} (\mathbb{R}v_2 + kv_1)$$

in  $\mathbb{C}$ . Then connect each  $nv_1 + mv_2 \in \Gamma_0$  to each point  $b_{k,n,m}$ , of  $\hat{B} \cap U_{n,m}$  by a simple path, say  $\gamma_{k,n,m}$  in  $U_{n,m}$  so that they are mutually disjoint. Then

$$\Gamma := \Gamma_0 \cup \bigcup_{k,n,m} \gamma_{k,n,m}$$

becomes a contractible tree connecting 0 with the end points, which are the critical values of  $\hat{f}$ .

We next remove from  $\Gamma$  all the end points  $b_{k,n,m}$  and denote the resulting space by

$$\Gamma' := \Gamma \setminus \bigcup_{k,n,m} \{b_{k,n,m}\} .$$

Finally for each of the removed end points  $b_{k,n,m}$  we fill in a small ball  $B_{k,n,m}$  centered at  $b_{k,n,m}$  and denote the resulting space by

$$\tilde{\Gamma} := \Gamma \cup \bigcup_{k,n,m} B_{k,n,m} = \Gamma' \cup \bigcup_{k,n,m} B_{k,n,m} .$$

We put

$$\begin{aligned}\hat{X}_\Gamma &= \hat{f}^{-1}(\Gamma) , \quad \hat{X}_{\Gamma'} = \hat{f}^{-1}(\Gamma') , \quad \hat{X}_{k,n,m} = \hat{f}^{-1}(B_{k,n,m}) , \\ \hat{X}_{\tilde{\Gamma}} &= \hat{f}^{-1}(\tilde{\Gamma}) , \quad F_{k,n,m} = \hat{f}^{-1}(b_{k,n,m}) .\end{aligned}$$

As  $\Gamma$  is tree, one can choose a neighbourhood  $U \subset \mathbb{C}$  of  $\Gamma$  such that  $\Gamma$  is a deformation retract of  $U$  and  $U$  is also a deformation retract of  $\mathbb{C}$ . Then we obtain a deformation retract from  $\hat{X}$  to  $\hat{X}_{\tilde{\Gamma}}$  and then to  $\hat{X}_\Gamma$ .

Combining this with Proposition 5.3, we obtain

$$H^{2n}(\hat{X}_{\tilde{\Gamma}}, \mathbb{Z}) \simeq H^{2n}(\hat{X}, \mathbb{Z}) \simeq \mathbb{Z}.$$

On the other hand,  $H^{2n}(\hat{X}_{\tilde{\Gamma}}, \mathbb{Z})$  can be also computed as follows. We notice that  $\Gamma'$  is contractible and that the fibre over  $\Gamma'$  is smooth and homeomorphic to  $F_0$ . Here  $F_0$  is the fibre over the base point 0. Thus

$$H^*(\hat{X}_{\tilde{\Gamma}'}, \mathbb{Z}) \simeq H^*(F_0, \mathbb{Z}) .$$

As each  $F_{k,n,m}$  is a deformation retract of  $\hat{X}_{k,n,m}$  and since all the  $\hat{X}_{k,n,m}$ 's are mutually disjoint ( $B_{k,n,m}$  being sufficiently small), it follows that

$$H^*(\bigcup_{k,n,m} \hat{X}_{k,n,m}, \mathbb{Z}) \simeq \bigoplus_{k,n,m} H^*(F_{k,n,m}, \mathbb{Z}) .$$

Moreover, since each  $B_{k,n,m} \cap \Gamma'$  is contractible and since the fibres over this set are homeomorphic to  $F_0$ , we also have

$$H^*(\hat{X}_{\tilde{\Gamma}'} \cap (\bigcup_{k,n,m} \hat{X}_{k,n,m}), \mathbb{Z}) \simeq \bigoplus_{k,n,m} H^*(F_0, \mathbb{Z}) .$$

Thus by the Mayer-Vietoris exact sequence we obtain

$$\begin{aligned}H^{2n}(\hat{X}_{\tilde{\Gamma}}, \mathbb{Z}) &= H^{2n}(\hat{X}_{\tilde{\Gamma}'} \cup (\bigcup_{k,n,m} \hat{X}_{k,n,m}), \mathbb{Z}) \\ &\simeq (H^{2n}(F_0, \mathbb{Z}) \oplus \bigoplus_{k,n,m} H^{2n}(F_{k,n,m}, \mathbb{Z})) / (\bigoplus_{k,n,m} H^{2n}(F_0, \mathbb{Z})) \\ &\simeq H^{2n}(F_0, \mathbb{Z}) \oplus \bigoplus_{k,n,m} (H^{2n}(F_{k,n,m}, \mathbb{Z}) / H^{2n}(F_0, \mathbb{Z})) .\end{aligned}$$

Since  $H^{2n}(\hat{X}_{\tilde{\Gamma}}, \mathbb{Z}) \simeq \mathbb{Z}$ , it follows that

$$H^{2n}(F_{k,n,m}, \mathbb{Z}) \simeq H^{2n}(F_0, \mathbb{Z})$$

for each singular fibre  $F_{k,n,m}$ . Since  $H^{2n}(F_0, \mathbb{Z}) \simeq \mathbb{Z}$ , it follows that

$$H^{2n}(F_{k,n,m}, \mathbb{Z}) \simeq \mathbb{Z} .$$

This implies the irreducibility of  $F_{k,n,m}$ , because  $F_{k,n,m}$  is a compact connected complex space of pure dimension  $n$  (hence of real dimension  $2n$ ) so that the rank of  $H^{2n}(F_{k,n,m}, \mathbb{Z})$  is the cardinality of the set of irreducible components of  $F_{k,n,m}$ .

Let  $y \in Y$  and  $Z = (X_y)_{\text{red}}$  be the reduction of the fibre  $X_y$  of the original fibration  $f$ . Now we know that  $Z$  is irreducible. Moreover, by the proof of Proposition 2.1, we also know that the image of the natural map

$$\pi_1(Z) \rightarrow \text{Ker}(\pi_1(X) \rightarrow \pi_1(Y)) \simeq \mathbb{Z}^{2n}$$

has finite cokernel. Thus the image is isomorphic to  $\mathbb{Z}^{2n}$  as well. Consequently we have a surjective homomorphism

$$\pi_1(Z) \rightarrow \mathbb{Z}^{2n} .$$

Since  $\mathbb{Z}^{2n}$  is isomorphic to the fundamental group of a real torus of dimension  $2n$ , say  $T^{2n}$ , the surjective morphism above is induced by a dominant continuous map

$$a : \pi_1(Z) \rightarrow \pi_1(T^{2n}) .$$

Since  $\pi_1(T^{2n})$  is commutative, this naturally induces a surjective morphism

$$a : H_1(Y, \mathbb{Z}) \rightarrow H_1(T^{2n}, \mathbb{Z}) .$$

Passing to the dual, we obtain an injective morphism

$$a^* : H^1(T^{2n}, \mathbb{Z}) \simeq \mathbb{Z}^{2n} \rightarrow H^1(Z, \mathbb{Z}) .$$

Let  $\nu : \tilde{Z} \rightarrow Z$  be a resolution of singularities of the complex space  $Z$  ([Hi77], main result). Then the composition of  $a$  and  $\nu$  defines a continuous map  $\tilde{a}$  such that its action on the first homology is the composition  $\tilde{a}_*$  of  $\nu_* : H_1(\tilde{Z}, \mathbb{Z}) \rightarrow H_1(Z, \mathbb{Z})$  with  $a_* : H_1(Z, \mathbb{Z}) \rightarrow H_1(T^{2n}, \mathbb{Z})$ .

Let  $\langle \varphi_i \rangle_{i=1}^{2n}$  be a basis of  $H^1(T^{2n}, \mathbb{Z})$  and consider their inverse images  $\tilde{a}^*(\varphi_i)$  as being represented by  $d$ -closed differential forms. Then the map given by integration

$$\tilde{Z} \ni x \mapsto \left( \int_{x_0}^x \tilde{a}^*(\varphi_i) \right)_{i=1}^{2n}$$

gives a differentiable map  $\rho : \tilde{Z} \rightarrow T^{2n}$  such that the induced morphism  $\rho_* : H_1(\tilde{Z}, \mathbb{Z}) \rightarrow H_1(T^{2n}, \mathbb{Z})$  is the homomorphism  $\tilde{a}_*$ .

Since we have isomorphisms

$$H^{2n}(\tilde{Z}, \mathbb{Z}) \cong H^{2n}(Z, \mathbb{Z}) \cong H^{2n}(T^{2n}, \mathbb{Z})$$

it follows that  $\tilde{a}$  is dominant and that we have a surjective homomorphism  $\pi_1(\tilde{Z}) \rightarrow \mathbb{Z}^{2n}$ . □

Let us return to the proof of Theorem 5.1 and recall that by Lemma 5.2 we may assume  $Y$  to be an elliptic curve. We need only to show that  $f$  is smooth; then Theorem 5.1 follows from Theorem 3.2.

So assume that a fibre  $Z$  of  $f$  is singular. We already know that  $Z$  is irreducible by Proposition 5.3. Denote by  $m$  the multiplicity of  $Z$  so that  $Z = mZ_{\text{red}}$ . Since  $X$  is smooth, hence  $Z_{\text{red}}$  is also a Cartier divisor on  $X$ . In particular, the dualizing sheaf  $\omega_{Z_{\text{red}}}$  is invertible. More precisely, by the adjunction formula and by  $K_X = \mathcal{O}_X$ , we have

$$\omega_{Z_{\text{red}}} = \mathcal{O}_X(Z_{\text{red}}) \otimes \mathcal{O}_{Z_{\text{red}}}$$

and therefore

$$\omega_{Z_{\text{red}}}^{\otimes m} \simeq \mathcal{O}_{Z_{\text{red}}} .$$

Since  $K_X = \mathcal{O}_X$ , the multiplicity  $m$  is nothing but the minimal positive integer satisfying this isomorphism (see eg. [BHPV04], p.111, Lemma 8.3).

Let  $\tilde{Z}$  be the minimal resolution of the normalization  $Z'$  of  $Z_{\text{red}}$ . Since  $Z_{\text{red}}$  is Gorenstein, the conductor ideal of  $Z' \rightarrow Z_{\text{red}}$  is of pure dimension 1 (if  $Z_{\text{red}}$  is not normal). Moreover, since  $\tilde{Z}$  is a minimal resolution, the canonical divisors of  $Z_{\text{red}}$



and  $\tilde{Z}$  differ by an effective divisor, called classically the subadjunction divisor. We conclude the well known formula

$$\omega_{\tilde{Z}}^{\otimes m} \simeq \mathcal{O}_{\tilde{Z}}(-D) ,$$

where  $D$  is an effective divisor, possibly 0 (if and only if  $Z_{\text{red}}$  is normal with at most rational double points as singularities).

Suppose first  $D = 0$ , hence  $\kappa(\tilde{Z}) = 0$ . Since  $\pi_1(\tilde{Z})$  maps onto  $\pi_1(T) \simeq \mathbb{Z}^4$  by Proposition 5.3, it follows again from surface classification

that  $\tilde{Z}$  is a complex torus of dimension 2. Since a complex torus of dimension 2 has no curve with negative self-intersection, it has no non-trivial crepant contraction to a normal surface  $Z'$ . Hence the three surfaces

$$\tilde{Z} , Z' , Z_{\text{red}}$$

are all isomorphic.

In particular,  $Z_{\text{red}}$  is also a smooth complex torus (of dimension 2) and  $\omega_{Z_{\text{red}}} \simeq \mathcal{O}_{Z_{\text{red}}}$ . This implies  $m = 1$  and  $Z = Z_{\text{red}}$  is smooth.

If  $D \neq 0$ , then  $\kappa(\tilde{Z}) = -\infty$ . From the classification of compact complex surfaces with  $\kappa = -\infty$  (see eg. [BHPV04], p.244, Table 10),  $\tilde{Z}$  is either birationally ruled, say over a curve  $C$ , or a surface of class *VII*. If it is birationally ruled, then  $H^1(\tilde{Z}, \mathbb{Z})$  is the pullback of  $H^1(C, \mathbb{Z})$ . This however implies that

$$\wedge^4 H^1(\tilde{Z}, \mathbb{Z}) = 0,$$

a contradiction to the proven injectivity of  $\wedge^4 H^1(T, \mathbb{Z}) \rightarrow \wedge^4 H^1(\tilde{Z}, \mathbb{Z})$ .

If  $\tilde{Z}$  is of class *VII*, then  $b_1(\tilde{Z}) = 1$ . This again contradicts the surjectivity of

$$\pi_1(\tilde{Z}) \rightarrow \pi_1(T) \simeq \mathbb{Z}^4$$

in Proposition 5.3.

Hence  $f : X \rightarrow Y$  is smooth.

Q.E.D. for Theorem 5.1.

Theorem 1.1 now follows from Theorems 2.3, 4.1 and 5.1.

## 6. THREEFOLDS WITHOUT MEROMORPHIC FUNCTIONS

Instead of assuming the existence of meromorphic functions we will require in this short concluding section the existence of some holomorphic vector fields or holomorphic 1-forms.

**Theorem 6.1.** *Let  $X$  be a smooth compact complex threefold which is homotopy equivalent to a torus. If the tangent bundle  $T_X$  is trivial, then  $X$  is biholomorphic to a torus.*

*Proof.* By our assumption (see for instance [Hu90], page 144)  $X \simeq G/\Gamma$ , where  $G$  is a simply connected complex 3-dimensional Lie group and  $\Gamma \simeq \pi_1(X)$  cocompact. By Lemma 6.2,  $G \simeq \mathbb{C}^3$  or  $G \simeq SL(2, \mathbb{C})$  as groups. But  $SL(2, \mathbb{C})$  is not contractible, e.g.

$$b_3(SL(2, \mathbb{C})) = 3$$

(see e.g. [Ko59]), hence  $X \simeq \mathbb{C}^3/\mathbb{Z}^6$ , and our claim follows. □

The paper [Ko59] was brought to our attention by I.Radloff.

For discussions concerning the first part of the following lemma, which is of course well-known to the experts, we would like to thank J.Winkelmann and in particular A.Huckleberry.

**Lemma 6.2.** *Let  $G$  be a simply connected 3-dimensional complex Lie group. Then*

- (1) *Either  $G \simeq \mathrm{SL}(2, \mathbb{C})$  as Lie group or  $G$  is solvable and  $G \simeq \mathbb{C}^3$  as complex manifold.*
- (2) *If  $G$  is solvable and if  $G$  contains a lattice  $\Gamma$  (i.e., a discrete subgroup such that  $G/\Gamma$  has bounded volume), such that  $\Gamma$  is abelian, then  $G$  is abelian and therefore  $G \simeq \mathbb{C}^3$  as Lie group.*

*Proof.* (1) By the Levi-Malcev decomposition,  $G$  is either semi-simple or solvable by reasons of dimension. In the semi-simple case,  $G \simeq \mathrm{SL}(2, \mathbb{C})$ . If  $G$  is solvable, then  $G \simeq \mathbb{C}^3$  (even in any dimension), see e.g. [Na75], prop.1.4.

(2) Since  $\Gamma$  is abelian, so does  $G$  by e.g. [Wi98] (3.14.6). Hence  $G \simeq \mathbb{C}^3$  as Lie group. □

**Lemma 6.3.** *Let  $X$  be a compact complex manifold with algebraic dimension  $a(X) = 0$ . Let  $V$  be a holomorphic rank  $r$  bundle on  $X$ : then the evaluation homomorphism*

$$ev : H^0(X, V) \otimes \mathcal{O}_X \rightarrow V$$

*is injective. In particular, all the global sections of  $V$  are carried by the trivial rank  $h$  subsheaf  $H^0(X, V) \otimes \mathcal{O}_X$ .*

*In particular,  $h^0(X, V) = h \leq r$  and, if  $h = r$  and  $\det(V) \cong \mathcal{O}_X$ , then  $V$  is trivial.*

*Proof.* For each point  $x \in X$  we have a linear map of  $\mathbb{C}$ -vector spaces

$$ev_x : H^0(X, V) \rightarrow V(x),$$

where  $V(x) := V_x / \mathfrak{M}_x V_x$  is the fibre of the vector bundle over the point  $x$  ( $\mathfrak{M}_x \subset \mathcal{O}_x$  being the maximal ideal).

We claim that  $ev_x$  is injective for a general point  $x \in X$ .

Otherwise, let  $m$  be the generic rank of  $ev_x$ : then we get a meromorphic map

$$k : X \dashrightarrow \mathrm{Grass}(h - m, H^0(X, V))$$

associating to  $x$  the subspace  $\ker(ev_x)$ .

By the projectivity of the Grassmann manifold,  $k$  must be constant. But a section vanishing at a general point is identically zero, which proves our assertion that  $H^0(X, V) \otimes \mathcal{O}_X$  yields a subsheaf of  $V$ .

Moreover, if  $h = r$ , the homomorphism  $ev$  induces a non constant homomorphism  $\Lambda^r(ev) : \mathcal{O}_X \rightarrow \det(V)$ . Thus, if  $\det(V)$  is trivial,  $\Lambda^r(ev)$  is invertible, hence  $ev$  is an isomorphism. □

**Corollary 6.4.** *Let  $X$  be a compact complex manifold of dimension  $n$  with algebraic dimension  $a(X) = 0$  and with trivial canonical divisor  $K_X$ . Then*

$$h^0(\Omega_X^1) \geq n \Leftrightarrow h^0(\Theta_X) \geq n \Leftrightarrow \Theta_X \cong \mathcal{O}_X^n \Leftrightarrow \Omega_X^1 \cong \mathcal{O}_X^n.$$

**Theorem 6.5.** *Let  $X$  be a smooth compact complex threefold with trivial canonical divisor  $K_X$  which is homotopy equivalent to a torus.*

*If  $h^0(X, \Omega_X^1) \geq 3$  or if  $h^0(X, T_X) \geq 3$ , then  $X$  is biholomorphic to a torus.*

*Proof.* By our Main Theorem 1.1 we may assume  $a(X) = 0$ , and applying the previous corollary 6.4 we get  $\Omega_X^1 \cong \mathcal{O}_X^3$ . Now we conclude by Theorem 6.1.  $\square$

**Remark 6.6.** It seems already difficult to exclude the case  $h^0(\Omega_X^1) = 2$ . Taking a basis  $\omega_1, \omega_2$ , we are able to exclude the case when both  $\omega_i$  are non-closed. Since  $X$  is not necessarily Kähler, the existence of a closed holomorphic 1-form does not lead to a non-trivial Albanese map, which is the obstacle to conclude.

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